Multifractal Analysis of inhomogeneous Bernoulli products

Athanasios BATAKIS and Benoît TESTUD

Abstract We are interested to the multifractal analysis of inhomogeneous Bernoulli products which are also known as coin tossing measures. We give conditions ensuring the validity of the multifractal formalism for such measures. On another hand, we show that these measures can have a dense set of phase transitions.

Keywords : Hausdorff dimension, multifractal analysis, Gibbs measure, phase transition.

1 Introduction

Let us consider the dyadic tree $\mathbb{T}$ (even though all the results in this paper can be easily generalised to any ℓ-adic structure, $\ell \in \mathbb{N}$), let $\Sigma = \{0,1\}^{\mathbb{N}}$ be its limit (Cantor) set and denote by $(\mathcal{F}_n)_{n\in\mathbb{N}}$ the associated filtration with the usual $0-1$ encoding.

For $\epsilon_1, ..., \epsilon_n \in \{0,1\}$ we denote by $I_{\epsilon_1...\epsilon_n}$ the cylinder of the nth generation defined by $I_{\epsilon_1...\epsilon_n} = \{x = (i_1, ..., i_n, i_{n+1}, ...) \in \Sigma \ ; \ i_1 = \epsilon_1, ..., i_n = \epsilon_n\}$. For every $x \in \Sigma$, $I_n(x)$ stands for the cylinder of $\mathcal{F}_n$ containing x.

If $(p_n)_n$ is a sequence of weights, $p_n \in (0,1)$, we are interested in Borel measures μ on Σ defined in the following way

$$\mu(I_{\epsilon_1...\epsilon_n}) = \prod_{j=1}^{n} p_j^{1-\epsilon_j}(1-p_j)^{\epsilon_j}. \tag{1}$$

A measure of this form will be referred to as an *inhomogeneous Bernoulli product*. The aim of this paper is to study multifractal properties of such measures.

The particular case where the sequence (p_n) is constant is well-known and provides an example of measure satisfying the multifractal formalism (see e.g [Fal97]). In the general case, Bisbas [Bis95] gave a sufficient condition on the sequence (p_n) ensuring

that μ is a multifractal measure (i.e. the level sets are not empty). However, the work of Bisbas does not provide the dimension of the level sets E_α associated to the measure μ.

Let us give a brief description of multifractal formalism. For a probability measure m on Σ, we define the *local dimension* (also called Hölder exponent) of m at $x \in \Sigma$ by

$$\alpha(x) = \liminf_{n\to+\infty} \alpha_n(x) = \liminf_{n\to+\infty} -\frac{\log m(I_n(x))}{n \log 2}.$$

The aim of multifractal analysis is to find the Hausdorff dimension, $\dim(E_\alpha)$, of the level set $E_\alpha = \{x : \alpha(x) = \alpha\}$ for $\alpha > 0$. The function $f(\alpha) = \dim(E_\alpha)$ is called the *singularity spectrum* (or multifractal spectrum) of m and we say that m is a *multifractal measure* when $f(\alpha) > 0$ for several $\alpha' s$.

The concepts underlying the multifractal decomposition of a measure go back to an early paper of Mandelbrot [Man74]. In the 80's multifractal measures were used by physicists to study various models arising from natural phenomena. In fully developped turbulence they were used by Frisch and Parisi [FP85] to investigate the intermittent behaviour in the regions of high vorticity. In dynamical system theory they were used by Benzi et al. [BPPV84] to measure how often a given region of the attractor is visited. In diffusion-limited aggregation (DLA) they were used by Meakin et al. [MCSW86] to describe the probability of a random walk landing to the neighborhood of a given site on the aggregate.

In order to determine the function $f(\alpha)$, Hentschel and Procaccia [HP83] used ideas based on Renyi entropies [Rén70] to introduce the generalized dimensions D_q defined by

$$D_q = \lim_{n\to+\infty} \frac{1}{q-1} \frac{\log\left(\sum_{I\in\mathcal{F}_n} m(I)^q\right)}{n \log 2},$$

(see also [GP83, Gra83]). From a physical and heuristical point of view, Halsey et al. [HJK$^+$86] showed that the singularity spectrum $f(\alpha)$ and the generalized dimensions D_q can be derived from each other. The Legendre transform turned out to be a useful tool linking $f(\alpha)$ and D_q. More precisely, it was suggested that

$$f(\alpha) = \dim(E_\alpha) = \tau^*(\alpha) = \inf(\alpha q + \tau(q),\ q \in \mathbb{R}), \tag{2}$$

where

$$\tau(q) = \limsup_{n\to+\infty} \tau_n(q) \quad \text{with} \quad \tau_n(q) = \frac{1}{n\log 2} \log\left(\sum_{I\in\mathcal{F}_n} m(I)^q\right).$$

(The sum runs over the cylinders I such that $m(I) \neq 0$.) The function $\tau(q)$ is called the L^q-spectrum of m and if the limit exists $\tau(q) = (q-1)D_q$.

Relation (2) is called the multifractal formalism and in many aspects it is analogous to the well-known thermodynamic formalism developed by Bowen [Bow75] and Ruelle [Rue78]. In general, the main problem is to obtain the minoration $\dim(E_\alpha) \geq \tau^*(\alpha)$.

For number of measures, this formalism can be verified rigorously. In particular, if the sequence (p_n) is constant or periodic, the measure μ given by (1) satisfies the multifractal formalism (e.g. [Fal97]). It is also the case for invariant measures in some dynamical systems (e.g [Col88, Fan94, Ran89]), for self-similars measures under separation conditions (e.g [CM92, Fen03, LN99, Ols95, Rie95, Ye05]) and for quasiindependent measures(e.g [BMP92, Heu98, Tes06a]).

Despite all these investigations mentioned, the exact range of the validity of the multifractal formalism is still not known. Olsen [Ols95] give a rigorous approach of multifractal formalism in a general context. This work and the paper of Brown, Michon and Peyrière [BMP92] enlighten the link between the minoration $\dim(E_\alpha) \geq \tau^*(\alpha)$ and the existence of auxiliary measures m_q (the so-called *Gibbs measure* [Mic83]) satisfying

$$\forall n,\ \forall I \in \mathcal{F}_n, \quad \frac{1}{C} m(I)^q 2^{-n\tau(q)} \leq m_q(I) \leq C m(I)^q 2^{-n\tau(q)},$$

where the constant $C > 0$ is independent of n and I. In fact, it is shown in [Ben94, BBH02] that the existence of a measure m_q satisfying

$$m_q(I) \leq C m(I)^q 2^{-n\tau(q)},$$

is sufficient to obtain the minoration $\dim(E_\alpha) \geq \tau^*(\alpha)$ for $\alpha = -\tau'(q)$. In this situation, the values of α for which the multifractal formalism may fail lie in intervals $(-\tau'(q^+), -\tau'(q^-))$ where q is a point of non differentiability of τ ($\tau'(q^+)$ and $\tau'(q^-)$ stands for the right and the left derivatives respectively). Such a point q will be called a *phase transition*.

If the weights p_n are not all the same, the measure μ defined by (1) is in general no shift-invariant and we cannot apply classical tools of ergodic theory, as Shannon-McMillan theorem (e.g [Bil65]), to get a lower bound of $\dim(E_\alpha)$ and the differentiability of the function τ.

Let us introduce the other following level sets defined by

$$\underline{E}_\alpha = \{x\ ;\ \alpha(x) \leq \alpha\},\ \overline{F}_\alpha = \left\{x\ ;\ \limsup_{n\to\infty} \alpha_n(x) \geq \alpha\right\},$$

and

$$F_\alpha = \left\{x\ ;\ \limsup_{n\to\infty} \alpha_n(x) = \alpha\right\}.$$

We can now state our main results. In section 2, we prove the following.

Theorem 1.1 *Let μ be an inhomogeneous Bernoulli product on Σ and $q \in \mathbb{R}$. We have*

$$\liminf_{n\to\infty} -q\tau_n'(q) + \tau_n(q) \le \dim\left(\underline{E}_{-\tau'(q^-)} \cap \overline{F}_{-\tau'(q^+)}\right) \le \sup\left\{\tau^*(-\tau'(q^+)), \tau^*(-\tau'(q^-))\right\}.$$

The proof of the lower bound relies on the construction of a special inhomogeneous Bernoulli product which has the dimension of the studied level set.

In section 3 we study the case $\alpha = -\tau'(q)$. The existence of $\tau'(q)$ is not sufficient to ensure the validity of multifractal formalism for such values of $\alpha's$. However, we prove that the multifractal formalism holds if the sequence $(\tau_n(q))$ converges. More precisely, we have

Theorem 1.2 *Suppose that the sequence $(\tau_n(q))$ converges at a point $q \in \mathbb{R}$. If $\tau'(q)$ exists and if $\alpha = -\tau'(q)$, we have*

$$\dim(E_\alpha \cap F_\alpha) = \tau^*(\alpha) = \alpha q + \tau(q). \tag{3}$$

We easily deduce the following

Corollary 1.3 *If p_n tend to p, as $n \to \infty$, then $\tau_\mu = \tau(p, .)$ and the mesure μ satisfies the multifractal formalism. Nevertheless, the measure μ can be singular with respect to the (homogeneous) Bernoulli measure associated to p.*

Theorem 1.2 leads us to study the differentiability of the L^q-spectrum $\tau(q)$. In section 4, we will see that the L^q-spectrum of an inhomogeneous Bernoulli product may be a very irregular function. In particular,

Theorem 1.4 *There exist inhomogeneous Bernoulli products presenting a dense set of phase transitions on $(1, +\infty)$.*

The are several examples of measures presenting phase transitions (see for instance [Tes06b] and the references therein). The example we propose in this work differs from previous ones at three points : first the phase transitions are situated at points $q > 1$ and not at negative ones, where constructions are easier to carry out. Secondly, the set of transitions is dense in $[1, \infty)$, that means as « bad » as can be. And finally, the measure presenting this pathologie is just a Bernoulli product ! Let us also point out that with some minor modifications our method can also apply to create a dense set of phase transitions within $(0, 1)$.

2 Proof of Theorem 1.1

We begin by a preliminary result.

Lemma 2.1 *If μ is an inhomogeneous Bernoulli product, then the functions $(\tau''_{\mu,n})$ are locally uniformly bounded on $(0,+\infty)$.*

Proof We denote by $\beta(p_i)$ the homogeneous Bernoulli measure of parameter p_i and by $\tau(p_i,q)$ it's τ function, $\tau(p_i,q)=\log_2(p_i^q+(1-p_i)^q)$. Using the fact that μ is the product of $\beta(p_i)$ we easily obtain

$$\tau_{\mu,n}(q)=\frac{1}{n}\sum_{i=1}^{n}\tau(p_i,q).$$

It is therefore sufficient to show that, for any $q_0>0$, there exists a constant $C=C(q_0)$ such that for all $p\in(0,1)$ and all $q>q_0$, $\dfrac{\partial^2\tau(p,q)}{\partial q^2}\le C$. We have

$$\begin{aligned}
\frac{\partial^2\tau(p,q)}{\partial q^2} &= \frac{p^q(\log_2 p)^2+(1-p)^q(\log_2(1-p))^2}{p^q+(1-p)^q}-\frac{\left(p^q\log_2 p+(1-p)^q\log_2(1-p)\right)^2}{(p^q+(1-p)^q)^2}\\
&= \frac{p^q(1-p)^q\left((\log_2 p)^2+(\log_2(1-p))^2-2\log_2 p\log_2(1-p)\right)}{(p^q+(1-p)^q)^2}\\
&= \frac{p^q(1-p)^q\left(\log_2\frac{p}{1-p}\right)^2}{(p^q+(1-p)^q)^2}\le[4p(1-p)]^q(\log_2\frac{p}{1-p})^2\\
&\le [4p(1-p)]^{q_0}(\log_2\frac{p}{1-p})^2,
\end{aligned}$$

which is uniformly bounded on $p\in(0,1)$ and the proof is complete. •

Lemma 2.1 allows us to give estimates for the lower and the upper Hausdorff dimension of the measure μ. They are respectively defined by

$$\dim_*(\mu)=\inf\{\dim(E),\ \ \mu(E)>0\}\,;\ \dim^*(\mu)=\inf\{\dim(E),\ \ \mu(E)=1\}.$$

We say that μ is exact if $\dim_*(\mu)=\dim^*(\mu)$ and we note $\dim(\mu)$ the common value. In the same way, we can define the lower and the upper Packing dimension Dim of the measure μ. It is well known that there exist some relations between these quantities and the derivatives of the function $\tau_\mu(q)$ at $q=1$. More precisely, it is proved in [Fan94, Heu98] that

$$-\tau'_\mu(1+)\le\dim_*(\mu)\le h_*(\mu)\le h^*(\mu)\le \operatorname{Dim}^*(\mu)\le-\tau'_\mu(1-),$$

where $h_*(\mu)$ and $h^*(\mu)$ stand for the lower and the upper entropy of the measure μ, defined as

$$h_*(\mu) = \liminf -\frac{1}{n\log 2}\sum_{I\in\mathcal{F}_n}\mu(I)\log\mu(I) = \liminf -\tau'_{\mu_n}(1)$$

and

$$h^*(\mu) = \limsup -\frac{1}{n\log 2}\sum_{I\in\mathcal{F}_n}\mu(I)\log\mu(I) = \limsup -\tau'_{\mu_n}(1).$$

By Lemma 2.1, we deduce (see [BH02, Heu98]) the following remark.

Remark 2.2 If μ is an inhomogeneous Bernoulli product then

$$\dim\mu = -\tau'_\mu(1^+) = h_*(\mu) = \liminf_{n\to\infty} -\tau'_{\mu_n}(1)$$

and

$$\mathrm{Dim}\,\mu = -\tau'_\mu(1^-) = h^*(\mu) = \limsup_{n\to\infty} -\tau'_{\mu_n}(1).$$

Fix $q\in\mathbb{R}$. To prove Theorem 1.1, we construct an auxiliary measure ν supported by the set $\underline{E}_{-\tau'(q^-)}\cap\overline{F}_{-\tau'(q^+)}$. More precisely, we consider a sequence of measures ν_n satisfying

$$\forall I\in\mathcal{F}_n,\quad \nu_n(I) = \frac{\mu(I)^q}{\sum_{I\in\mathcal{F}_n}\mu(I)^q} = \mu(I)^q|I|^{\tau_{\mu,n}(q)}. \tag{4}$$

($|I| = 2^{-n}$ stands for the diameter of I). The following lemma implies that the sequence (ν_n) converges in the weak* sense to a probability measure ν which is by construction an inhomogeneous Bernoulli product.

Lemma 2.3 *Let $n\in\mathbb{N}$ and $I\in\mathcal{F}_n$. If μ is an inhomogeneous Bernoulli product, we have $\nu_n(I) = \nu_{n+1}(I)$.*

Proof Take $n>0$ and $I\in\mathcal{F}_n$. We can compute

$$\nu_{n+1}(I) = \frac{\sum_{J\in\mathcal{F}_1}\mu(IJ)^q}{\sum_{I\in\mathcal{F}_n}\sum_{J\in\mathcal{F}_1}\mu(IJ)^q} = \frac{\mu(I)^q(p_{n+1}^q + (1-p_{n+1})^q)}{\sum_{I\in\mathcal{F}_n}(p_{n+1}^q + (1-p_{n+1})^q)\mu(I)^q}$$

and therefore $\nu_{n+1}(I) = \nu_n(I)$ for all $I\in\mathcal{F}_n$. •

By remark 2.2, we then deduce that the Hausdorff and the Packing dimension of ν are given by an entropy formula. In other terms, we have

$$\dim \nu = \liminf_{n\to\infty} -\tau'_{\nu,n}(1) = h_*(\nu)$$

and

$$\mathrm{Dim}\, \nu = \limsup_{n\to\infty} -\tau'_{\nu,n}(1) = h^*(\nu).$$

Now we can prove Theorem 1.1.

Proof of Theorem 1.1 The upper bound is a well known fact of multifractal formalism (see for instance [BMP92]). In fact we have

1. If $\alpha \le -\tau'(0^+)$ then $\dim E_\alpha \le \dim \underline{E}_\alpha \le \tau^*(\alpha)$.
2. If $\alpha \ge -\tau'(0^-)$ then $\dim F_\alpha \le \dim \overline{F}_\alpha \le \tau^*(\alpha)$.
3. $-\tau'(0^+) \le \alpha \le -\tau'(0^-)$ then $\tau^*(\alpha) = \tau(0) = 1$ and the upper bound follows.

Relation (4) easily gives $\tau_{\nu,n}(s) = \tau_{\mu,n}(qs) - s\tau_{\mu,n}(q)$. From remark 2.2, using the inhomogeneous Bernoulli property of μ and ν, we deduce that

$$-\tau'_\nu(1^+) = \liminf -\tau'_{\nu,n}(1) = \liminf \left(-q\tau'_{\mu,n}(q) + \tau_{\mu,n}(q)\right).$$

The following lemma then implies the lower bound.

Lemma 2.4 *We have* $\nu\left(\underline{E}_{-\tau'(q^-)} \cap \overline{F}_{-\tau'(q^+)}\right) = 1$.

Remark 2.5 Contrary to more regular situations (e.g [BBH02, Heu98, Ols95]), we cannot obtain the more precise result $\nu\left(\overline{E}_{-\tau'(q^-)} \cap \underline{F}_{-\tau'(q^+)}\right) = 1$ where

$$\overline{E}_\alpha = \{x\ ;\ \alpha(x) \ge \alpha\},\ \underline{F}_\alpha = \left\{x\ ;\ \limsup_{n\to\infty} \alpha_n(x) \le \alpha\right\}.$$

Proof of Lemma 2.4 For $\eta > 0$ we put $\beta = -\tau'_\mu(q^-) + \eta$ and we prove that $\nu(\Sigma \setminus \underline{E}_\beta) = 0$. In a similar way, it can be shown that $\nu(\Sigma \setminus \overline{F}_\gamma) = 0$ for $\gamma < -\tau'_\mu(q^+)$. The lemma then easily follows.

It suffices to show that $\Sigma \setminus \underline{E}_\beta = \left\{x \in \Sigma\ ;\ \liminf_{n\to\infty} \alpha_n(x) > \beta\right\}$ is of 0 ν-measure. Consider the collection $\mathcal{R}_n(\beta)$ of cylinders $I \in \mathcal{F}_n$ satisfying $\dfrac{\log \mu(I)}{\log |I|} > \beta$. It is clear that $\Sigma \setminus \underline{E}_\beta \subset \liminf_{n\to\infty} \tilde{\mathcal{R}}_n(\beta)$ with $\tilde{\mathcal{R}}_n(\beta) = \{x \in \Sigma\ ;\ I_n(x) \in \mathcal{R}_n(\beta)\}$.

Let $(\tau_{\mu,n_k})_{k\in\mathbb{N}}$ be the subsequence of $(\tau_{\mu,n})_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty}\tau_{\mu,n_k}(q)=\tau_\mu(q)$. Using the convergence of $\tau_{\mu,n_k}(q)$ we can choose (and fix) $t<0$ such that for k big enough

$$\tau_\mu(q+t)-\tau_{\mu,n_k}(q)<-\left(\beta-\frac{\eta}{2}\right)t=\left(\tau'_\mu(q^-)-\frac{\eta}{2}\right)t.$$

Since $\mu(I)^{-t}|I|^{\beta t}\le 1$ if $I\in\mathcal{R}_n(\beta)$, we have

$$\begin{aligned}
\nu(\tilde{\mathcal{R}}_{n_k}(\beta))=\sum_{I\in\mathcal{R}_{n_k}(\beta)}\nu(I) &= \sum_{I\in\mathcal{R}_{n_k}(\beta)}\mu(I)^q|I|^{\tau_{\mu,n_k}(q)}=\sum_{I\in\mathcal{R}_{n_k}(\beta)}\mu(I)^{q+t}|I|^{\tau_{\mu,n_k}(q)-\beta t}\mu(I)^{-t}|I|^{\beta t}\\
&\le \sum_{I\in\mathcal{R}_{n_k}(\beta)}\mu(I)^{q+t}|I|^{\tau_{\mu,n_k}(q)-\beta t}\le |I|^{-\frac{\eta}{4}t}\sum_{I\in\mathcal{F}_{n_k}}\mu(I)^{q+t}|I|^{\tau_\mu(q+t)-\frac{\eta}{4}t}\\
&\le |I|^{-\frac{\eta}{4}t}\sum_{I\in\mathcal{F}_{n_k}}\mu(I)^{q+t}|I|^{\tau_{\mu,n_k}(q+t)}=|I|^{-\frac{\eta}{4}t}.
\end{aligned}$$

For the last inequality, we used the fact that $\tau_\mu(q+t)=\limsup\tau_{\mu,n}(q+t)$. We deduce that

$$\liminf_{n\to\infty}\nu(\tilde{\mathcal{R}}_n(\beta))=0$$

and the lemma easily follows.

•

The proof of Theorem 1.1 is now completed. •

3 Proof of Theorem 1.2

We will use the following result.

Proposition 3.1 *For $q\in\mathbb{R}$, let (τ_{μ,n_k}) be the subsequence of $(\tau_{\mu,n})$ such that*

$$\lim_{k\to\infty}\tau_{\mu,n_k}(q)=\limsup_{n\to\infty}\tau_{\mu,n}(q)=\tau_\mu(q).$$

Then, we have

$$\tau'_\mu(q^-)\le\liminf_{k\to\infty}\tau'_{\mu,n_k}(q)\le\limsup_{k\to\infty}\tau'_{\mu,n_k}(q)\le\tau'_\mu(q^+)$$

where $\tau'_\mu(q^-)$ and $\tau'_\mu(q^+)$ stand for the left and the right hand dérivative of τ_μ at q.

Hence, if $\tau'_\mu(q)$ exists, we have

$$\lim_{k\to\infty}\tau'_{\mu,n_k}=\tau'_\mu(q).$$

Proof We only prove the inequality $\limsup_{k\to\infty} \tau'_{\mu,n_k}(q) \le \tau'_\mu(q^+)$. The proof of $\tau'_\mu(q^-) \le \liminf_{k\to\infty} \tau'_{\mu,n_k}(q)$ is similar.

Take $\epsilon > 0$ and $\tilde{q} > q$ such that

$$\left|\frac{\tau_\mu(\tilde{q}) - \tau_\mu(q)}{\tilde{q} - q} - \tau'_\mu(q^+)\right| < \epsilon/3.$$

We can chose k big enough to have

$$\frac{|\tau_{\mu,n_k}(q) - \tau_\mu(q)|}{|\tilde{q} - q|} \quad < \quad \epsilon/3$$

and

$$\tau_{\mu,n_k}(\tilde{q}) \quad \le \quad \tau_\mu(\tilde{q}) + (\tilde{q} - q)\epsilon/3.$$

We then obtain

$$\begin{aligned}
\tau'_\mu(q^+) \quad &\ge \quad \frac{\tau_\mu(\tilde{q}) - \tau_\mu(q)}{\tilde{q} - q} - \epsilon/3 \\
&= \quad \frac{\tau_\mu(\tilde{q}) - \tau_{\mu,n_k}(\tilde{q}) + \tau_{\mu,n_k}(\tilde{q}) - \tau_{\mu,n_k}(q) + \tau_{\mu,n_k}(q) - \tau_\mu(q)}{\tilde{q} - q} - \epsilon/3 \\
&\ge \quad -\epsilon/3 + \tau'_{\mu,n_k}(q) - \epsilon/3 - \epsilon/3 = \tau'_{\mu,n_k}(q) - \epsilon
\end{aligned}$$

and the proof easily follows. •

We can now prove Theorem 1.2.

Proof of Theorem 1.2. Let ν be the Gibbs-measure defined in Lemma 2.3. Since

$$\tau_{\nu,n}(s) = \tau_{\mu,n}(qs) - s\tau_{\mu,n}(q)$$

we get

$$\tau'_{\nu,n}(1) = q\tau'_{\mu,n}(q) - \tau_{\mu,n}(q).$$

Using the convergence of $\tau_{\mu,n}(q)$ we deduce from Proposition 3.1 that

$$\lim_{n\to\infty} \tau'_{\nu,n}(1) = \lim_{n\to\infty} \left(q\tau'_{\mu,n}(q) - \tau_{\mu,n}(q)\right) = q\tau'_\mu(q) - \tau_\mu(q).$$

Since ν is also an inhomogeneous Bernoulli product, we deduce from remark 2.2 that $\tau'_\nu(1)$ exists and

$$\dim \nu = \operatorname{Dim} \nu = -\tau'_\nu(1) = -q\tau'_\mu(q) + \tau_\mu(q).$$

On the other hand, for $I \in \mathcal{F}_n$, we have

$$\frac{\log \nu(I)}{\log |I|} = q \frac{\log \mu(I)}{\log |I|} + \tau_{\mu,n}(q).$$

Since

$$\lim_{n\to\infty} \frac{\log \nu(I_n(x))}{\log |I_n(x)|} = \dim \nu = \operatorname{Dim} \nu \;\; ; \nu\text{-a.s.}$$

we obtain that $\lim_{n\to\infty} \dfrac{\log \mu(I_n(x))}{\log |I_n(x)|} = -\tau'_\mu(q)$, ν-a.s. We conclude that

$$\dim (E_\alpha \cap F_\alpha) \geq \dim \nu = \tau^*_\mu(\alpha).$$

The opposite inequality being always valid, the proof is done. •

We end this section with a few comments about Theorem 1.2.

As mentionned in the introduction of the paper, the validity of the multifractal formalism is often easier to obtain for the values of α that can be written $\alpha = -\tau'(q)$. However, the following example shows that there exist inhomogeneous Bernoulli products that do not satisfy the multifractal formalism even at their differentiability points $\alpha = -\tau'(q)$. Thus, the convergence of the sequence $(\tau_n(q))$ is really necessary for the validity of the multifractal formalism in our context.

To see that, let $(n_k)_{k\geq 1}$ be a sequence of integers such that

$$n_1 = 1, \quad n_k < n_{k+1} \quad \text{and} \quad \lim_{k\to+\infty} \frac{n_{k+1}}{n_k} = +\infty,$$

and consider the inhomogeneous Bernoulli product μ given by the sequence (p_n) such that

$$p_i = p \quad \text{if} \quad n_{2n-1} \leq i < n_{2n} \quad \text{and} \quad p_i = \tilde{p} \quad \text{if} \quad n_{2n} \leq i < n_{2n+1},$$

with $0 < p < \tilde{p} < 1/2$.

The calculation of the function τ is classical. By observing that

$$\mu(I_{\epsilon_1\ldots\epsilon_n 0})^q + \mu(I_{\epsilon_1\ldots\epsilon_n 1})^q = [(p_{n+1}^q + (1-p_{n+1})^q]\mu(I_{\epsilon_1\ldots\epsilon_n})^q,$$

we easily deduce that

$$\sum_{I\in\mathcal{F}_n} \mu(I)^q = \prod_{k=1}^n [p_k^q + (1-p_k)^q].$$

Then, if N_n is the number of integer $k \leq n$ such that $p_k = p$, we have

$$\tau_n(q) = \frac{N_n}{n} \log_2(p^q + (1-p)^q) + (1 - \frac{N_n}{n}) \log_2(\tilde{p}^q + (1-\tilde{p})^q).$$

Since that $\liminf_n \frac{N_n}{n} = 0$ and $\limsup_n \frac{N_n}{n} = 1$, we get

$$\tau(q) = \sup(\log_2(p^q + (1-p)^q), \log_2(\tilde{p}^q + (1-\tilde{p})^q).$$

So, except for $q = 0$ and $q = 1$, $\tau'(q)$ exists. Moreover,

$$\forall I \in \mathcal{F}_n, \quad \mu(I) \geq \mu(I_{00\cdots 0}) = p^{N_n} \tilde{p}^{n-N_n}.$$

Thus,

$$\forall I \in \mathcal{F}_n, \quad -\frac{\log_2(\mu(I))}{n} \leq \frac{N_n}{n}(-\log_2 p) + (1 - \frac{N_n}{n})(-\log_2 \tilde{p}),$$

and we have

$$\forall I \in \mathcal{F}_n, \quad \liminf_n -\frac{\log_2(\mu(I))}{n} \leq \inf(-\log_2 p, -\log_2 \tilde{p}) = -\log_2 \tilde{p}.$$

Finally, if $-\log_2 \tilde{p} < \alpha = -\tau'(q) < -\log_2 p$, we have $E_{-\tau'(q)} = \emptyset$ and the multifractal formalism is not satisfied for a such α.

Moreover, this example shows that the function τ may be not differentiable at the positive values of q. Therefore, the situation differs from this one obtained by Heurteaux [Heu98] for quasi-Bernoulli measure for which τ is differentiable on $\mathbb{R}$. It also differs from this one obtained by Testud in [Tes06b] for weak quasi-Bernoulli measure for which the phase transitions only appears for $q < 0$.

In fact, the function τ of an inhomogeneous Bernoulli product may be very irregular. This is the object following section.

4 Proof of Theorem 1.4

From now, we denote par $\tau(p, .)$ the τ function of the homogeneous Bernoulli product of parameter p. Moreover, whenever we use the notation p_i for a weight in $(0, 1)$ we will also note $\tau_i = \tau(p_i, .)$.

Before the proof of Theorem 1.4, we present a few lemmas.

Lemma 4.1 *For any $p_1 < p_2 < p_3$ in $(0, 1/2)$ consider the functions $\tau_1 = \tau(p_1, .), \tau_2 = \tau(p_2, .)$ and $\tau_3 = \tau(p_3, .)$. We have that $\dfrac{\tau_1 - \tau_2}{\tau_2 - \tau_3}$ is decreasing on $(1, +\infty)$.*

Although the proof only uses elementary calculus, it is a litte bit "tricky" and cannot be omitted.

Proof of Lemma 4.1 Taking into account the trivial equality

$$\tau(p', q) - \tau(p'', q) = \int_{p''}^{p'} \frac{\partial \tau}{\partial p}(p, q) dp$$

we only need to show that if $p' < p''$ then $\dfrac{\partial \tau}{\partial p}(p', q) : \dfrac{\partial \tau}{\partial p}(p'', q)$ is decreasing on $q \in (1, \infty)$. We get

$$\begin{aligned} \frac{\partial \tau}{\partial p}(p', q) : \frac{\partial \tau}{\partial p}(p'', q) &= \frac{1}{p'} \frac{1 - (-1 + 1/p')^{q-1}}{1 + (-1 + 1/p')^q} : \frac{1}{p''} \frac{1 - (-1 + 1/p'')^{q-1}}{1 + (-1 + 1/p'')^q} \\ &= p'' \frac{1 - s_1{}^{q-1}}{1 + s_1{}^q} : p' \frac{1 - s_2{}^{q-1}}{1 + s_2{}^q} \end{aligned}$$

where $s_1 = -1 + 1/p' > 1$ and $s_2 = -1 + 1/p'' > 1$.

If we set $f(s, q) = \ln \dfrac{1 - s^{q-1}}{1 + s^q}$, with $s, q > 1$, it is sufficient to prove that $\dfrac{\partial f}{\partial s} f(s, q)$ is decreasing in q. We calculate

$$\frac{\partial f}{\partial s} f(s, q) = \frac{(q-1)s^{q-2}}{s^{q-1} - 1} - \frac{q s^{q-1}}{s^q + 1}.$$

By multiplying by s, we need to show that $\dfrac{(q-1)s^{q-1}}{s^{q-1} - 1} - \dfrac{q s^q}{s^q + 1}$ is decreasing which is equivalent to $q - 1 + \frac{q-1}{s^{q-1}-1} - q + \frac{q}{s^q+1}$ being decreasing.

Put $Q = q - 1$; it remains to show that $\dfrac{q-1}{s^{q-1} - 1} + \dfrac{q}{s^q + 1} = \dfrac{Q}{s^Q - 1} + \dfrac{Q}{s^{Q+1} + 1} + \dfrac{1}{s^{Q+1} + 1}$ decreases in $Q > 0$. The last term being decreasing it suffices to show that $\dfrac{Q}{s^Q - 1} + \dfrac{Q}{s^{Q+1} + 1}$ is doing the same. By taking derivatives we need to show that

$$s^{Q+1}(s^Q - 1 - s^Q \ln s^Q) + s^Q - 1 - \ln s^Q \leq 0.$$

Since, $(s^Q - 1 - s^Q \ln s^Q) < 0$, it suffices to show that

$$s^Q(s^Q - 1 - s^Q \ln s^Q) + s^Q - 1 - \ln s^Q = s^{2Q} - s^{2Q} \ln s^Q - \ln s^Q - 1 = g(s^Q) \leq 0$$

where $g(x) = x^2 - x^2 \ln x - \ln x - 1$. Moreover, the sign of $g'(x) = x - x \ln x^2 - 1/x$ is the same of the sign of $x^2 - x^2 \ln x^2 - 1$ if $x > 1$. Since, $y - 1 \leq y \ln y$ for $y > 1$, we deduce that g is decreasing on $(1, +\infty)$. By observing than $g(1) = 0$, we obtain that $\dfrac{\partial f}{\partial s} f(s, q)$ is decreasing on $(1, +\infty)$ and the Lemma 4.1 is proved. •

Lemma 4.2 *Take $\tau = \lambda\tau(p_1,.) + (1-\lambda)\tau(p_2,.)$ with $0 < p_1 < p_2 < 1/2$ and $\lambda \in (0,1)$. For $p_0 \in (0,1/2)$ one of the following occurs :*

1. *either $\tau(q) \neq \tau(p_0,q)$, for all $q > 1$,*
2. *either there exists $q_0 > 1$ such that $\tau(q) > \tau(p_0,q)$ for $1 < q < q_0$ and $\tau(q) < \tau(p,q)$ for $q > q_0$. In this case q_0 is then the unique point of $(1,+\infty)$ for which $\tau(q) = \tau(p_0,q)$.*

Proof of Lemma 4.2. Let us first remark that τ and $\tau(p_0,.)$ can coincide at one point only if $p_0 \in (p_1,p_2)$. Moreover, $\tau(q) = \tau(p_0,q)$ implies

$$\frac{\tau(p_1,q) - \tau(p_0,q)}{\tau(p_0,q) - \tau(p_2,q)} = \frac{1-\lambda}{\lambda}.$$

By Lemma 4.1 this can only occur once and Lemma 4.2 easily follows on the decreasing property of the ratio. •

Lemma 4.3 *Take $\lambda_1, \lambda_2 \in (0,1)$ such that $\lambda_1 + \lambda_2 = 1$, $1 < p_1 < p_2 < 1/2$ and set $\tau = \lambda_1\tau_1 + \lambda_2\tau_2$. Fix $1 < q_1 < q_2 < +\infty$ and consider $p_1 < p_4 < p_2 < p_5 < 1/2$ such that $\tau(p_4,q_1) = \tau(q_1)$. Then there is a unique convex combination $\tilde{\tau}$ of τ_1, τ_4 and τ_5 such that*

$$\tilde{\tau}(q_1) = \tau(q_1) \text{ and } \tilde{\tau}(q_2) = \tau(q_2).$$

Furthermore, for $i = 1,2$, we have $\tau'(q_i) \neq \tilde{\tau}'(q_i)$ and $\tau(q) \neq \tilde{\tau}(q)$ if $1 < q \neq q_i$.

Proof of Lemma 4.3.

First note that it is easy to see that there exists $p_4 \in (p_1,p_2)$ such that $\tau(p_4,q_1) = \tau(q_1)$.

It then suffices to show that the linear system

$$\left\{\begin{array}{llllll} \lambda_3\tau_1(q_1) & +\lambda_4\tau_4(q_1) & +\lambda_5\tau_5(q_1) & = & \tau(q_1) \\ \lambda_3\tau_1(q_2) & +\lambda_4\tau_4(q_2) & +\lambda_5\tau_5(q_2) & = & \tau(q_2) \\ \lambda_3 & +\lambda_4 & +\lambda_5 & = & 1 \end{array}\right. \qquad (\mathbf{S})$$

has a unique positive solution $(\lambda_3,\lambda_4,\lambda_5)$. The existence of a unique solution is easy to verify. Let us show that this solution is positive.

First note that $\lambda_4 \neq 1$. Indeed, if $\lambda_4 = 1$, since $\tau(q_1) = \tau_4(q_1)$, we have $\lambda_3(\tau_1(q_1) - \tau_5(q_1)) = 0$. Thus, $\lambda_3 = \lambda_5 = 0$ and $\tau(q_2) = \tau_4(q_2)$ which is not possible by Lemma 4.2.

Therefore, since $\tau(q_1) = \tau_4(q_1)$, the first equation of the system gives that

$$\frac{\lambda_3}{\lambda_3+\lambda_5}\tau_1(q_1)+\frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5(q_1)=\lambda_1\tau_1(q_1)+\lambda_2\tau_2(q_1)\in(\tau_5(q_1),\tau_1(q_1)) \tag{5}$$

This implies that $\frac{\lambda_3}{\lambda_3+\lambda_5}\in(0,1)$. We deduce that $\lambda_3\lambda_5>0$. Moreover, since $\tau_5<\tau_2$, we also have $\frac{\lambda_3}{\lambda_3+\lambda_5}>\lambda_1$.

Let us show that λ_3 and λ_5 are positive. Otherwise, by the above remark, we have $\lambda_3<0$, $\lambda_5<0$ and $\lambda_4>0$. By the system (S) we have

$$\tau_4(q)=\frac{\lambda_1-\lambda_3}{\lambda_4}\tau_1(q)+\frac{\lambda_2}{\lambda_4}\tau_2(q)-\frac{\lambda_5}{\lambda_4}\tau_5(q)$$

at the points $q=q_1$ and $q=q_2$. We then obtain that

$$\frac{\lambda_1-\lambda_3}{\lambda_4}\frac{\tau_1-\tau_4}{\tau_4-\tau_2}(q)=\frac{\lambda_2}{\lambda_4}-\frac{\lambda_5}{\lambda_4}\frac{\tau_4-\tau_5}{\tau_4-\tau_2}(q)$$

for $q=q_1$ and $q=q_2$. Since $p_1<p_4<p_2$, by Lemma 4.1 the function $\frac{\tau_1-\tau_4}{\tau_4-\tau_2}$ is decreasing. On the other hand, since $p_4<p_2<p_5$, Lemma 4.1 implies that the function $\frac{\tau_4-\tau_5}{\tau_4-\tau_2}=1+\frac{\tau_2-\tau_5}{\tau_4-\tau_2}$ is increasing. Thus, these functions cannot coincide at two points so we conclude that λ_3 and λ_5 are positive.

Let us now prove that $\lambda_4>0$. By (5) we have

$$\frac{\lambda_3}{\lambda_3+\lambda_5}\tau_1(q_1)+\frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5(q_1)=\lambda_1\tau_1(q_1)+\lambda_2\tau_2(q_1)$$

which gives that

$$\lambda_2\tau_2(q_1)=\left(\frac{\lambda_3}{\lambda_3+\lambda_5}-\lambda_1\right)\tau_1(q_1)+\frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5(q_1).$$

Using Lemma 4.1, for $q>q_1$ we get

$$\lambda_2\tau_2(q)>\left(\frac{\lambda_3}{\lambda_3+\lambda_5}-\lambda_1\right)\tau_1(q)+\frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5(q)$$

and

$$\lambda_1\tau_1(q)+\lambda_2\tau_2(q)>\frac{\lambda_3}{\lambda_3+\lambda_5}\tau_1(q)+\frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5(q).$$

In particular, for $q=q_2$ we find that

$$\lambda_3\tau_1(q_2)+\lambda_5\tau_5(q_2)+\lambda_4\tau(q_2)<\tau(q_2)=\lambda_3\tau_1(q_2)+\lambda_4\tau_4(q_2)+\lambda_5\tau_5(q_2)$$

and we deduce that

$$\lambda_4 \tau(q_2) < \lambda_4 \tau_4(q_2).$$

Since $\tau(q_1) = \tau_4(q_1)$, it follows from Lemma 4.1 that $\lambda_4 > 0$.

The last assertion follows directly from the independancy of the vector families

$$\left\{ \begin{pmatrix} \tau_1(q_1) \\ \tau_4(q_1) \\ \tau_5(q_1) \end{pmatrix}, \begin{pmatrix} \tau_1(q_2) \\ \tau_4(q_2) \\ \tau_5(q_2) \end{pmatrix}, \begin{pmatrix} \tau_1'(q_i) \\ \tau_4'(q_i) \\ \tau_5'(q_i) \end{pmatrix} \right\}$$

and

$$\left\{ \begin{pmatrix} \tau_1(q_1) \\ \tau_4(q_1) \\ \tau_5(q_1) \end{pmatrix}, \begin{pmatrix} \tau_1(q_2) \\ \tau_4(q_2) \\ \tau_5(q_2) \end{pmatrix}, \begin{pmatrix} \tau_1(q) \\ \tau_4(q) \\ \tau_5(q) \end{pmatrix} \right\},$$

which can be easily established. •

Remark 4.4 In the proof of Lemma 4.3 it is clear that when p_5 is close to p_2, the solution of the system (**S**) converges to $(\lambda_1, 0, \lambda_2)$ and $\tilde{\tau}$ converges to τ.

The following result generalizes Lemma 4.3 for any convex combination of functions $\tau(p_i, .)$.

Lemma 4.5 *Let τ be a convex combination of functions $\tau(p_i, .)$ where $0 < p_i \leq 1/2$, $i = 1, ..., n \geq 2$. For any $1 < q_1 < q_2 < \infty$ there exists another convex combination $\tilde{\tau}$ of functions $\tau(p_j', .)$ such that*

- *$\tilde{\tau}(q_i) = \tau(q_i)$ and $\tilde{\tau}'(q_i) \neq \tau'(q_i)$, $i = 1, 2$,*
- *for $q \notin \{q_1, q_2\}$, $\tilde{\tau}(q) \neq \tau(q)$.*

Proof of Lemma 4.5.

The case $n = 2$ is given by Lemma 4.3. The case $n > 2$ is easy to derive. Suppose $\tau = \sum_{k=1}^{n} \lambda_k \tau(p_k, .)$ and let $\tau_1 = \tau(p_1, .)$ and $\tau_2 = \tau(p_2, .)$ be the first two functions of the convex combination. By Lemma 4.3 there exists a convex combination $\hat{\tau}$ of three $\tau(p, .)$ functions such that

1. $\frac{1}{\lambda_1+\lambda_2} \left(\lambda_1 \tau_1(q_i) + \lambda_2 \tau_2(q_i) \right) = \hat{\tau}(q_i)$, for $i = 1, 2$
2. $\frac{1}{\lambda_1+\lambda_2} \left(\lambda_1 \tau_1'(q_i) + \lambda_2 \tau_2'(q_i) \right) \neq \hat{\tau}'(q_i)$ for $i = 1, 2$.

The function $\tilde{\tau} = (\lambda_1+\lambda_2)\hat{\tau}+\sum_{k=3}^{n} \lambda_k \tau(p_k, .)$ satisfies then the conclusion of Lemma 4.5. •

The following lemma is easy and the proof is left to the reader.

Lemma 4.6 *For any $p_1, ..., p_n$ and any convex combination τ of $\tau(p_1, .), ..., \tau(p_n, .)$ there exists an inhomogeneous Bernoulli measure μ whose multifractal spectrum equals τ.*

We can now prove Theorem 1.4.

Proof of Theorem 1.4

In fact, and in order to avoid technicalities, we only prove the following easier version of Theorem 1.4 and then indicate the changes needed to extend the proof in the general case.

Theorem 4.7 *There exists an inhomogeneous Bernoulli product μ such that the spectrum τ of μ has an infinite set of the phase transitions on $(1, +\infty)$. Moreover, this set has a finite point of accumulation.*

Proof The strategy of the demonstration is the following : we first find inhomogeneous Bernoulli products that are not derivable at a finite number of predefined points and we construct the measure μ using Cantor's diagonal argument.

Fix $(q_n)_{n\geq 1}$ a sequence of real numbers nested in the sense that $1 < q_1 < ... < q_{2n-1} < q_{2n+1} < q_{2n+2} < q_{2n} < ... < q_2$ for all $n \geq 1$ and $\cap_n (q_{2n+1}, q_{2n+2}) = \{q_0\}$. In particular, $\lim q_n = q_0$. Let $p_1, p_2 \in (0, 1/2)$ and consider $\tau_1 = \frac{1}{2}\tau(p_1, .) + \frac{1}{2}\tau(p_2, .)$. By Lemma 4.6 we can construct a Bernoulli product μ_1 of spectrum τ_1. Then, Lemma 4.5 implies the existence of a convex combination τ_2 of $\tau(p_i, .)$'s functions, such that

$$\tau_1(q_i) = \tau_2(q_i) \text{ , for } i = 1, 2 \text{ and } \tau_1'(q_i) \neq \tau_2'(q_i).$$

We can define a measure μ_2 of spectrum τ_2. Using μ_1 and μ_2, we can construct a measure ν_2 of spectrum $\rho_2 = \max\{\tau_1, \tau_2\}$. To do that, we take a sequence of integers $(\ell_k)_k$ such that $\dfrac{\ell_{k+1}}{\sum_1^k \ell_i} \to \infty$. On dyadique intervals of length between $2^{-\ell_{2k}}$ and $2^{-\ell_{2k+1}}$ we apply the weight distribution of μ_1 and on dyadique intervals of length between $2^{-\ell_{2k+1}}$ and $2^{-\ell_{2k+2}}$ we apply the weight distribution of μ_2, where $k \in \mathbb{N}$. It is easy to verify that the resulting inhomogeneous measure ν_2 has spectrum $\rho_2 = \max\{\tau_1, \tau_2\}$. The spectrum of ν_2 is not differentiable at q_1 and q_2.

We proceed by induction to construct a measure ν_n which has a non differentiable spectrum for points $q_1, \cdots, q_{2n-2}$. Suppose the measures $\nu_1 = \mu_1$, μ_2, ν_2,..., μ_n, ν_n constructed and denote by $\rho_n = \max\{\tau_1, ...\tau_n\}$ where τ_i is the spectrum of the measure μ_i, $i \in \{1..., n\}$. Let us construct μ_{n+1} and ν_{n+1}.

One of the following two cases hold :

Case 1 Lemma 4.5 provides a function τ_{n+1} satisfying :

1. $\tau_{n+1}(q_{2n-i}) = \rho_n(q_{2n-i})$ for $i = 0, 1$ and $\tau_{n+1}(q) \neq \rho_n(q)$ if $q \notin \{q_{2n-1}, q_{2n}\}$,
2. $\tau'_{n+1}(q_{2n-1}) > \rho'_n(q_{2n-1})$, $\tau'_{n+1}(q_{2n}) < \rho'_n(q_{2n})$

Therefore we have $\tau_{n+1} > \rho_n$ on (q_{2n-1}, q_{2n}) and $\tau_{n+1} < \rho_n$ on $(1, \infty) \setminus [q_{2n-1}, q_{2n}]$. Let μ_{n+1} be the inhomogeneous Bernoulli measure of spectrum τ_{n+1}. To define the measure ν_{n+1} we use the previous procedure convenably adapted : Take $(\ell_k)_k$ a sequence of integers such that $\dfrac{\ell_{k+1}}{\sum_1^k \ell_i} \to \infty$. On dyadique intervals of length between $2^{-\ell_{(n+1)k+i}}$ and $2^{-\ell_{(n+1)k+i+1}}$ apply the weight distribution of μ_i, where $i = 1, ..., n+1$ and $k \in \mathbb{N}$. It is easy to verify that the resulting inhomogeneous measure ν_{n+1} has spectrum $\rho_{n+1} = \max\{\tau_1, ...\tau_{n+1}\}$ on $(1, \infty)$. Remark that this spectrum equals τ_{n+1} on $[q_{2n-1}, q_{2n}]$ and $\rho_n = \max\{\tau_1, ...\tau_n\}$ elsewhere on $[1, \infty)$. Clearly, in this case, the function $\rho_{n+1} = \max(\tau, \tau_{n+1})$ is not differentiable at $q_1, \cdots, q_{2n}$.

Case 2 Lemma 4.5 provides for all choices of $p_5 > p_2$ a function τ_{n+1} satisfying :

1. $\tau_{n+1}(q_{2n-i}) = \rho_n(q_{2n-i})$, for $i = 0, 1$ and $\tau_{n+1}(q) \neq \rho_n(q)$ if $q \notin \{q_{2n-1}, q_{2n}\}$,
2. $\tau'_{n+1}(q_{2n-1}) < \rho'_n(q_{2n-1})$, $\tau'_{n+1}(q_{2n}) > \rho'_n(q_{2n})$

In this case,

$$\tau_{n+1} < \rho_n \text{ on } (q_{2n-1}, q_{2n}) \text{ and } \tau_{n+1} > \rho_n \text{ on } (q_{2n-3}, q_{2n-1}) \cup (q_{2n}, q_{2n-2}).$$

The function $\tilde{\rho}_{n+1} = \max(\rho_n, \tau_{n+1})$ is not differentiable at q_{2n-1} and q_{2n} but we lose the phase transitions q_{2n-3}, q_{2n-2} and we don't know what happens for the other phase transitions $q_1, \cdots, q_{2n-4}$.
To avoid this problem we use remark 4.4. From this, when p_5 converges to p_2, τ_{n+1} converges to the convex combination T of $\tau(p_i, .)$ functions which is equal to ρ_n on (q_{2n-3}, q_{2n-2}). Since T differs from ρ_n on (q_{2n-5}, q_{2n-3}) and (q_{2n-2}, q_{2n-4}), we can choose p_5 sufficiently close to p_2 such that

$$\tau_{n+1}\left(\frac{q_{2n-3} + q_{2n-5}}{2}\right) < \rho_n\left(\frac{q_{2n-3} + q_{2n-5}}{2}\right)$$

and

$$\tau_{n+1}\left(\frac{q_{2n-2} + q_{2n-4}}{2}\right) < \rho_n\left(\frac{q_{2n-2} + q_{2n-4}}{2}\right).$$

We deduce that there exist $q_{2n-5} < q' < q_{2n-3}$ and $q_{2n-2} < q'' < q_{2n-4}$ such that $\tau_{n+1} = \rho_n$ at q' and q'' and $\tau_{n+1} < \rho_n$ on (q_1, q') and (q'', q_2).

The modified family of $\tilde{q}_i$'s defined by

$$\tilde{q}_i = \begin{cases} q_i & \text{if } i \notin \{2n-3, 2n-2\} \\ q' & \text{if } i = 2n-3 \\ q'' & \text{if } i = 2n-2 \end{cases}$$

have the same properties as the initial q_i's. Moreover, $\rho_{n+1} = \max(\rho_n, \tau_{n+1})$ is not differentiable at points $q_1, \cdots, q_{2n}$. We proceed as above for the construction of the measures μ_{n+1} and ν_{n+1} which have spectra τ_{n+1} and ρ_{n+1} respectively.

To end the proof we use Cantor's diagonal argument : take $(\ell_k)_k$ as before and define the measure ν by applying the weight distribution of ν_k on dyadique intervals of length between $2^{-\ell_k}$ and $2^{-\ell_{k+1}}$. The spectrum of the measure ν equals then $\tau = \sup_{n\in\mathbb{N}} \rho_n = \sup_{n\in\mathbb{N}} \tau_n$. By construction, the set of non-derivability points of the function τ is infinite and has q_0 as accumulation point. •

Remark 4.8 The second case of the proof of Theorem 4.7 seems to be inexistent (in our numerical simulations) but we have not been able to prove that only the first case arises.

Let us now give some hints concerning the proof of Theorem 1.4.

Fix $(q_n)_n$ a sequence of real numbers, dense in $[1, \infty)$ and nested in the sense that $q_{2n+1} < q_{2n+2}$ and $\{q_1, ..., q_{2n}\} \cap [q_{2n+1} - \frac{1}{2^n}, q_{2n+2} + \frac{1}{2^n}] = \emptyset$ for all $n \geq 0$. We can then follow the proof of Theorem 4.7 until case 2, the first case being carried out exactly in the same way.

The second case has to be slightly modified. The technical, but not difficult, part is to ensure that the modified q_i's still form a dense subset of $[1, \infty)$ and that the difference of the left and right derivative at the q_i's does not go to 0. To do that we take p_5 sufficiently close to p_2 (in the construction of τ_{n+1}) to have :

- $|q_i - \tilde{q}_i| < \frac{1}{2^n} \inf_{1\leq j<j'\leq 2n+2} |q_j - q_{j'}|$
- $|\delta_n(q_i) - \delta_{n+1}(\tilde{q}_i)| < \frac{1}{2^n}\delta_n(q_i)$,

where $\delta_n(q_i)$ stands for the difference between the right and left derivative at q_i of $\sup_{1\leq k\leq n} \tau_k$. The proof of Theorem 1.4 is then completed in the same way as above. •

Références

[BBH02] F. BenNasr, I. Bhouri, and Y. Heurteaux. The validity of the multifractal formalism : results and examples. *Advances in Mathematics*, **165** : 264–284, 2002.

[Ben94] F. BenNasr. Analyse multifractale de mesures. *C. R Acad. Sci. Paris Sér . I Math.*, **319** : 807–810, 1994.

[BH02] A. Batakis and Y. Heurteaux. On relations between entropy and Hausdorff dimension of measures. *Asian Journal of Mathematics*, **6** (3) : 399–408, 2002.

[Bil65] P. Billingsley. *Ergodic Theory and Information*. John Wiley & Sons, 1965.

[Bis95] A. Bisbas. A multifractal analysis of an interesting class of measures. *Colloq. Math.*, **69** : 37–42, 1995.

[BMP92] G. Brown, G. Michon, and J. Peyrière. On the Multifractal Analysis of Measures. *J. Stat. Phys.*, **66** : 775–790, 1992.

[Bow75] R. Bowen. Equilibrium states and the ergodic theory of Anosov diffeomorphisms. In *Lecture Notes in Mathematics*, volume **470** . Springer-Verlag New York-Berlin, 1975.

[BPPV84] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani. On the multifractal nature of fully developed turbulence and chaotic system. *J. Phys. A*, **17** : 3521–3531, 1984.

[CM92] R. Cawley and R. D. Mauldin. Multifractal decompositions of Moran fractals. *Advances in Mathematics*, **92** : 196–236, 1992.

[Col88] P. Collet. Hausdorff dimension of singularities for invariant measures of expanding dynamical systems. In *Dynamical Systems Valparaiso 1986*, volume **1331** of *Lecture Notes in Mathematics*. R Bamon et al, Springer-Verlag Berlin, 1988.

[Fal97] K. Falconer. *Techniques in Fractal Geometry*. John Wiley & Sons Ltd., New-York, 1997.

[Fan94] A.H. Fan. Sur la dimension des mesures. *Studia Math.*, **111** : 1–17, 1994.

[Fen03] D.J. Feng. Smothness of the ℓ^q spectrum of self similar meausres with overlaps. *J. london Math. Soc.*, **68** : 102–118, 2003.

[FP85] U. Frisch and G. Parisi. On the singularity structure of fully developed turbulence. In *Proc. Internat. School Phys. Enrico Fermi*, pages 84–88. U Frisch (North-Holland, Amsterdam), 1985.

[GP83] P. Grassberger and I. Procaccia. Characterization of strange sets. *Phys. Rev. Lett.*, **50** : 346–349, 1983.

[Gra83] P. Grassberger. Generalized dimension of strange attractors. *Phys. Lett. A*, **97** : 227–230, 1983.

[Heu98] Y. Heurteaux. Estimations de la dimension inférieure et de la dimension supérieure des mesures. *Ann. Inst. H. Poincaré Probab. Statist.*, **34** : 309–338, 1998.

[HJK+86] T. C. Halsey, M. H. Jensen, L.P. Kadanoff, I. Procaccia, and B. Shraiman. Fractal measures and their singularities : The characterization of strange sets. *Phys. Rev. A*, **33** : 1141–1151, 1986.

[HP83] H. Hentschel and I. Procaccia. The infinite number of generalized dimensions of fractals and strange attractors. *Physica D*, **8** : 435–444, 1983.

[LN99] K.S. Lau, , and S. M. Ngai. Multifractal measures and a weak separation condition. *Adv. Math.*, **141** : 45–96, 1999.

[Man74] B.B. Mandelbrot. Intermittent turbulence in self-similar cascades : Divergence of high moments and dimension of the carrier. *J. Fluid Mech.*, **62** : 331–358, 1974.

[MCSW86] P. Meakin, A. Conoglio, H. Stanley, and T. Witten. Scaling properties for the surfaces of fractal and nonfractal objects : An infinite hierarchy of critical exponents. *Phys. Rev. A*, **34** : 3325–3340, 1986.

[Mic83] G. Michon. Mesures de Gibbs sur les Cantor réguliers. *Ann. Inst. H. Poincaré Phys. Théor.*, **58** : 267–285, 1983.

[Ols95] L. Olsen. A multifractal formalism. *Advances in Mathematics*, **116** : 82–196, 1995.

[Ran89] D. Rand. The singularity spectrum $f(\alpha)$ for cookie-cutter. *Ergod. Theory Dynam. Syst.*, **9** : 527–541, 1989.

[Rén70] A. Rényi. *Probability Theory*. North-Holland, Amsterdam, 1970.

[Rie95] R. Riedi. An improved multifractal formalism and self-similar measures. *J. Math. Anal. Appl.*, **189** : 462–490, 1995.

[Rue78] D. Ruelle. *Thermodynamic Formalism*. Addison-Wesley, Reading, MA, 1978.

[Tes06a] B. Testud. Mesures quasi-bernoulli au sens faible : résultats et exemples. *Ann. Inst. H. Poincaré Probab. Statist.*, **42** : 1–35, 2006.

[Tes06b] B. Testud. Phase transitions for the multifractal analysis of self-similar measures. *Nonlinearity*, **19** : 1201–1217, 2006.

[Ye05] Y. L. Ye. Multifractal of self-conformal measures. *Nonlinearity*, **18** : 2111–2133, 2005.